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Olimpiada Naţională de Matematică
Etapa locală, 10 februarie 2024
Clasa a IX – a

V1

BAREM ORIENTATIV de CORECTARE şi NOTARE:

Problema 1: soluție orientativă

$M \in (AB) \Rightarrow \overrightarrow{AM}$ și $\overrightarrow{AB}$ coliniari $\Rightarrow \overrightarrow{AM} = x * \overrightarrow{AB}, x \in \mathbb{R}$
$N \in (AC) \Rightarrow \overrightarrow{AN}$ și $\overrightarrow{AC}$ coliniari $\Rightarrow \overrightarrow{AN} = y * \overrightarrow{AC}, y \in \mathbb{R}$ **1p**

$\overrightarrow{NC} = \overrightarrow{NA} + \overrightarrow{AC} = -y * \overrightarrow{AC} + \overrightarrow{AC} = (1-y) * \overrightarrow{AC}$
$\overrightarrow{MB} = \overrightarrow{MA} + \overrightarrow{AB} = -x * \overrightarrow{AB} + \overrightarrow{AB} = (1-x) * \overrightarrow{AB}$ **1p**

$\overrightarrow{MD} = \overrightarrow{MB} + \overrightarrow{BD} = (1-x)\overrightarrow{AB} + \overrightarrow{BD} = (1-x)\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = (1-x)\overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) = (1 - x - \frac{1}{2}) * \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} \Rightarrow \overrightarrow{MD} = \frac{1-2x}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$ **1p**
$\overrightarrow{ND} = \overrightarrow{NC} + \overrightarrow{CD} = (1-y)\overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} = (1-y)\overrightarrow{AC} + \frac{1}{2}(\overrightarrow{CA} + \overrightarrow{AB}) = (1 - y - \frac{1}{2}) * \overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB} \Rightarrow$
$\overrightarrow{MD} = \frac{1-2y}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB}$ **1p**
$\overrightarrow{MB} + \overrightarrow{MD} = (1-x)\overrightarrow{AB} + \frac{1-2x}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} = (\frac{3}{2} - 2x)\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$ **1p**
$\overrightarrow{NC} + \overrightarrow{ND} = (1-y)\overrightarrow{AC} + \frac{1-2y}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB} = (\frac{3}{2} - 2y)\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB}$ **1p**

$\boldsymbol{\overrightarrow{MB} + \overrightarrow{MD}}$ ***și*** $\boldsymbol{\overrightarrow{NC} + \overrightarrow{ND}}$ ***sunt coliniari*** $\Leftrightarrow \frac{\frac{3}{2}-2x}{\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}-2y} \Rightarrow (\frac{3}{2} - 2x)(\frac{3}{2} - 2y) = \frac{1}{4}$ **1p**

$$\Rightarrow (3-4x)(3-4y) = 1$$

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Problema 2: soluție orientativă

$x = card(A \cap B)$

$\Rightarrow x \leq y < 2^y$ **1p**

$y = card(A \cup B)$

Dacă $x = 1 \Rightarrow \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24}$ nu are soluție

Dacă $x = 2 \Rightarrow \frac{1}{2} + \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} \Rightarrow \frac{1}{y} + \frac{1}{2^y} = \frac{7}{24}$

$y = 2 \Rightarrow \frac{1}{2} + \frac{1}{4} = \frac{7}{24} \Leftrightarrow \frac{3}{4} < \frac{7}{24}$

$y = 3 \Rightarrow \frac{1}{3} + \frac{1}{8} = \frac{11}{24} > \frac{7}{24}$

Pentru $y > 3 \Rightarrow \frac{1}{y} < \frac{1}{3}$

$\Rightarrow \frac{1}{y} + \frac{1}{2^y} < \frac{1}{3} + \frac{1}{8} < \frac{11}{24}$

$2^y > 2^3 \Rightarrow \frac{1}{2^y} < \frac{1}{8}$ **2p**

Dacă $x = 3 \Rightarrow \frac{1}{3} + \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} \Rightarrow \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} - \frac{1}{3} = \frac{11}{24}$

Pentru $y = 3 \Rightarrow \frac{1}{3} + \frac{1}{8} = \frac{11}{24} \Rightarrow y < 3\ convine\ pentru\ x = 3$

Pentry $y > 3 \Rightarrow \frac{1}{y} < \frac{1}{3}$

$\Rightarrow \frac{1}{y} + \frac{1}{2^y} < \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$

$2^y > 2^3 \Rightarrow \frac{1}{2^y} < \frac{1}{8}$

Pentru $y > 3$ ecuația nu are soluții. Analog pt $y < 3$ ecuația nu are soluții.

1p

Dacă $x = 4 \Rightarrow \frac{1}{4} + \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} \Rightarrow \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} - \frac{1}{4} = \frac{13}{24}$

$y = 4 \Rightarrow \frac{1}{4} + \frac{1}{16} = \frac{5}{16} < \frac{13}{24}$; Înlocuind $y < 4$ nu se verifică.

$y \geq 5 \Rightarrow \frac{1}{y} \leq \frac{1}{5}$

$\Rightarrow \frac{1}{y} + \frac{1}{2^y} \leq \frac{1}{5} + \frac{1}{32} = \frac{37}{160} < \frac{13}{24}$

$2^y > 2^5 \Rightarrow \frac{1}{2^y} \leq \frac{1}{2^5}$ **1p**

Dacă $x \geq 4 \Rightarrow \frac{1}{x} \leq \frac{1}{4}$

$4 \leq x \leq y \Rightarrow 4 \leq y \Rightarrow \frac{1}{4} \geq \frac{1}{y} \Rightarrow \frac{1}{y} \leq \frac{1}{4}$

$2^4 \leq 2^y$

$$\frac{1}{2^4} \geq \frac{1}{2^y} \Leftrightarrow \frac{1}{2^y} \leq \frac{1}{2^4}$$

$$\frac{1}{x} \leq \frac{1}{4}$$

$\frac{1}{y} \leq \frac{1}{4}$ $\quad \Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{2^y} \leq \frac{1}{4}+\frac{1}{4}+\frac{1}{2^4}=\frac{9}{16}<\frac{19}{24}$

$\frac{1}{2^y} \leq \frac{1}{2^4}$ **1p**

Soluția convenabila $x = 3$ și $y = 3 \Rightarrow card\,(A \cup B) = card\,(A \cap B) \Rightarrow card(A) = card(B)$. **1p**

Problema 3: soluție orientativă

$$\left\{\frac{4x+3}{3x-2}\right\}=\frac{4x+3}{3x-2}-\left[\frac{4x+3}{3x-2}\right]=\frac{2x-2}{3x-2}$$

1p

$\left[\frac{4x+3}{3x-2}\right]=\frac{4x+3}{3x-2}-\frac{2x-2}{3x-2}=\frac{2x+5}{3x-2}$

1p

$$\frac{2x+5}{3x-2}=k, k\in\mathbb{Z};$$

$$2x+5=k\cdot(3x-2)\Rightarrow 2x+5-3kx+2k=0\Rightarrow x(2-3k)\ +5+2k=0\Rightarrow$$

$\Rightarrow x=\frac{2k+5}{3k-2}$

1p

$$\left[\frac{4\frac{2k+5}{3k-2}+3}{3\frac{2k+5}{3k-2}-2}\right]=k\Rightarrow\left[\frac{\frac{8k+20+9k-6}{3k-2}}{\frac{6k+15-6k+4}{3k-2}}\right]=k\Rightarrow\left[\frac{\frac{17k+14}{3k-2}}{\frac{19}{3k-2}}\right]=k\Rightarrow\left[\frac{17k+14}{19}\right]=k\Rightarrow$$

1p

$$\Rightarrow k\le\frac{17k+14}{19}<k+1$$

1p

$$19k\le 17k+14\Rightarrow 2k\le 14\Rightarrow k\le 7$$

$$\Rightarrow k\in(\tfrac{-5}{2},7]$$

$$17k+14<19k+19\Rightarrow -2k+5\Rightarrow k>\frac{-5}{2}$$

1p

$$k\in\mathbb{Z}\Rightarrow k\in\{-2,-1,0,1,2,3,4,5,6,7\}$$

$$x\in\left\{\frac{-1}{8},\frac{-3}{5},\frac{-5}{2},7,\frac{9}{4},\frac{11}{7},\frac{13}{10},\frac{15}{13},\frac{17}{16},1\right\}$$

1p

Problema 4: soluție orientativă

$$\frac{bc}{b+c}=\frac{2abc}{2a(b+c)},\ \frac{ab}{a+b}=\frac{2abc}{2c(a+b)},\ \frac{ac}{a+c}=\frac{2abc}{2b(a+c)}$$

1p

Aplicăm inegalitatea mediilor pentru $2a$ și $b+c$ și obținem:

$$\frac{bc}{b+c}=\frac{2abc}{2a(b+c)}\geq\frac{2abc}{\left(\frac{2a+b+c}{2}\right)^2}=\frac{8abc}{(2a+b+c)^2}$$ **1p**

Scriem omoloagele:

$$\frac{ab}{a+b}=\frac{2abc}{2c(a+b)}\geq\frac{2abc}{\left(\frac{2c+b+a}{2}\right)^2}=\frac{8abc}{(a+b+2c)^2}$$

$$\frac{ac}{a+c}=\frac{2abc}{2b(a+c)}\geq\frac{2abc}{\left(\frac{2b+a+c}{2}\right)^2}=\frac{8abc}{(a+2b+c)^2}$$

Insumăm inegalitățile și obținem:

$$\frac{bc}{b+c}+\frac{ab}{a+b}+\frac{ac}{a+c}\geq 8abc\left(\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2}\right)$$ **1p**

Aplicăm inegalitatea:

$3(x^2+y^2+z^2)\geq(x+y+z)^2$

pentru $x=\frac{1}{2a+b+c}, y=\frac{1}{a+b+2c}, z=\frac{1}{a+2b+c}$ **1p**

$$\frac{bc}{b+c}+\frac{ac}{a+c}+\frac{ab}{a+b}\geq 8abc\left(\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2}\right)$$

$$\geq 8abc\,\frac{1}{3}\left(\frac{1}{2a+b+c}+\frac{1}{a+2b+c}+\frac{1}{a+b+2c}\right)^2$$ **1p**

Aplicăm inegalitatea *Titu Andreescu* pentru:

$$a_1=1, a_2=1, a_3=1$$
$$x_1=2a+b+c$$
$$x_2=a+2b+c$$
$$x_3=a+b+2c$$

1p

Și obținem:

$$8abc\,\frac{1}{3}\left(\frac{1^2}{2a+b+c}+\frac{1^2}{a+2b+c}+\frac{1^2}{a+b+2c}\right)^2\geq 8abc\,\frac{1}{3}\left[\frac{(1+1+1)^2}{4(a+b+c)}\right]^2=$$

$$=8abc\,\frac{1}{3}\left(\frac{3^2}{4(a+b+c)}\right)^2=8abc\cdot\frac{1}{3}\cdot\frac{81}{16}=\frac{27abc}{2}$$

1p

Notă: Orice altă soluţie corectă se punctează corespunzător